ON THE EFFECT OF SHEAR DEFORMATION AND ROTATORY INERTIA IN VIBRATIONS OF BEAMS ON ELASTIC FOUNDATION

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The Timoshenko-type equation is derived for vibration of a homogeneous beam on an elastic Winkler foundation. An approximate solution of this equation is given for the case of an infinite beam subjected to an instantaneously applied force. The solution obtained becomes the solution for a free beam if the coefficient of elasticity of the foundation is assumed equal to zero. Furthermore, by differentiation with respect to time, the solution for an instantaneous force gives the solution for an impulse (Dirac-type), and by the use of the Duhamel integral the solutions for an impulse of finite duration and for a variable force can be obtained.

The differential equations of motion for a beam on a Winkler foundation, with shear deformation and rotatory inertia taken into account, are of the form

$$EI\,\frac{\partial^3 y_i}{\partial x^3} + k'FG\,\frac{\partial y_c}{\partial x} - I\rho\,\frac{\partial^3 y_i}{\partial x\,\partial t^2} = 0$$

$$\rho F\,\frac{\partial^2 y}{\partial t^2} - k'FG\,\frac{\partial^2 y_s}{\partial x^2} + ky = q \qquad (1)$$

$$\frac{\partial y}{\partial x} = \frac{\partial y_i}{\partial x} + \frac{\partial y_s}{\partial x}$$

Here, y is the total deflection of the beam; y_i is the deflection caused by bending; y_s is the deflection caused by shear; ρ is the density of the material of the beam; I, F are the moment of inertia and the area of the cross-section; E, G are the Young modulus and the shear modulus; k is the coefficient of the foundation; q is the lateral loading; k' is a coefficient depending on the form of the cross-section.

The system (1) can be reduced to the following equation:

$$EI\frac{\partial^4 y}{\partial x^4}-\frac{EIk}{k'FG}\frac{\partial^2 y}{\partial x^2}+\left(\rho F+\frac{k\rho I}{k'FG}\right)\frac{\partial^2 y}{\partial t^2}-\left(\frac{\rho EI}{k'G}+\rho I\right)\frac{\partial^4 y}{\partial x^2\,\partial t^2}+ + \frac{\rho^2 I}{k'G}\frac{\partial^4 y}{\partial t^4}+ky=q-\frac{EI}{k'FG}\frac{\partial^2 q}{\partial x^2}+\frac{\rho I}{k'FG}\frac{\partial^2 q}{\partial t^2} \tag{2}$$

Noting that

$$M(x,t)=-EI\frac{\partial^2 y_i}{\partial x^2},\qquad Q(x,t)=k'FG\frac{\partial y_s}{\partial x} \tag{3}$$

for the case of a concentrated force at $x = 0$, we have the following boundary conditions:

$$\frac{\partial y_i}{\partial x}(0,t)=0,\qquad \frac{\partial^3 y_i}{\partial x^3}(0,t)=\frac{P}{2EI},\qquad \frac{\partial y_s}{\partial x}(0,t)=-\frac{P}{2k'FG} \tag{4}$$

The last condition will be derived from the equation of equilibrium of the beam with shear deformation only

$$k'FG\frac{\partial^2 y_s}{\partial x^2}-\rho F\frac{\partial^2 y_s}{\partial t^2}-ky_s=0 \tag{5}$$

Hence

$$\frac{\partial^3 y_s}{\partial x^3}(0,t)=-\frac{Pk}{2(k'FG)^2} \tag{6}$$

We have then

$$\frac{\partial y}{\partial x}(0,t)=-\frac{P}{2k'FG},\qquad \frac{\partial^3 y}{\partial x^3}(0,t)=\frac{P}{2EI}-\frac{Pk}{2(k'FG)^2}. \tag{7}$$

In addition, the function y and its derivatives should be zero for $x\to\infty$. The initial conditions are

$$y(x,0)=0,\ \frac{\partial y}{\partial t}(x,0)=0 \tag{8}$$

Applying the Fourier cosine transformation

$$Y(\xi,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty y\cos\xi x\,dx$$

to Equation (2), we obtain the following equation:

$$\left(\frac{c_1c_2c_4}{c_3}\right)^2\frac{\partial^4 Y}{\partial t^4}+\left[c_1^2+c_2^2c_4^2+\xi^2\left(c_2^2+\frac{c_1^2c_4^2}{c_3^2}\right)\right]\frac{\partial^2 Y}{\partial t^2}+ + (c_3^2+c_4^2\xi^2+\xi^4)Y=\sqrt{\frac{2}{\pi}}\frac{Pc_5^2}{2}\left(1+\xi^2\frac{c_4^2}{c_3^2}\right) \tag{9}$$

where

$$c_1^2 = \frac{\rho F}{EI}, \qquad c_2^2 = \frac{\rho}{E}, \qquad c_3^2 = \frac{k}{EI}, \qquad c_4^2 = \frac{k}{k'FG}, \qquad c_5^2 = \frac{1}{EI}$$

In the characteristic equation for (9), the term with the fourth power results in a quantity, in the lower part of the frequency spectrum, which is smaller by one order as compared to the small correction caused by shear and rotatory inertia. This fact is known in the literature. For a simple beam, it was noted by Timoshenko [1], who showed that the fourth derivative with respect to time corresponds to a small quantity of the second order in the frequency equation. In our case, the characteristic equation for (9) is identical with the frequency equation of free vibration. Therefore, neglecting the term with the fourth derivative with respect to time, we obtain

$$Y(\xi, t) = \sqrt{\frac{2}{\pi}} \frac{Pc_5^2(1 + c_4^2 c_3^{-2}\xi^2)}{2(c_3^2 + c_4^2\xi^2 + \xi^4)} (1 - \cos\omega t) \tag{10}$$

where

$$\omega = \sqrt{\frac{c_3^2 + c_4^2\xi^2 + \xi^4}{c_1^2 + c_2^2 c_4^2 + \xi^2(c_2^2 + c_1^2 c_4^2 c_3^{-2})}}$$

From the inversion theorem

$$y(x, t) = \frac{Pc_5^2}{\pi} \int_0^\infty \frac{\cos\xi x\,(1 + c_4^2 c_3^{-2}\xi^2)(1 - \cos\omega t)\,d\xi}{c_3^2 + c_4^2\xi^2 + \xi^4} \tag{11}$$

Using the dimensionless time $\tau = c_3 t/c_1$ and substituting $\xi^2 = c_3 z$, we write (11) in the form

$$y(x, \tau) = \frac{Pc_5^2}{2\pi c_3^{3/2}} \int_0^\infty \frac{\cos(x\sqrt{c_3 z})(1 + c_4^2 c_3^{-1} z)(1 - \cos\omega_1\tau)}{\sqrt{z}(1 + c_4^2 c_3^{-1} z + z^2)}\,dz \tag{12}$$

where

$$\omega_1 = \sqrt{\frac{1 + c_4^2 c_3^{-1} z + z^2}{1 + c_2^2 c_4^2 c_1^{-2} + z(c_4^2 c_3^{-1} + c_2^2 c_3 c_1^{-2})}}$$

If Expression (12) is decomposed into two integrals, the first of them gives the static deflection of the beam on a Winkler foundation subjected to a concentrated force P, including the effect of shear.

For a beam with rectangular cross-section, assuming $k' = 0.833$, $k = k_0 b$, $G = 0.375\,E$, we obtain

$$\frac{c_4^2}{c_3} = \gamma, \quad \frac{c_4^2 c_2^2}{c_1^2} = 3.04\,\gamma^2, \quad \frac{c_2^2 c_3}{c_1^2} = 0.305\,\gamma \qquad \left(\gamma = 0.935\sqrt{\frac{k_0 h}{E}}\right) \tag{13}$$

Here, k_0 is the coefficient of foundation per unit area of contact; h and b are the depth and the width of the beam, respectively; γ is a dimensionless parameter determining the effect of shear and rotatory inertia.

In the majority of practical cases of beams on soil foundation, γ is a small quantity as compared to 1 and its square power may be neglected. Consequently

$$y(x,\tau) = \frac{Pc_5^2}{2\pi c_3^{3/2}} \int_0^\infty \frac{\cos(x\sqrt{c_3 z})(1+\gamma z)(1-\cos\omega_2\tau)}{\sqrt{z}(1+\gamma z+z^2)}\,dz \tag{14}$$

where

$$\omega_2 = \sqrt{\frac{1+\gamma z+z^2}{1+1.305\,\gamma z}}$$

Thus, the effect of shear deformation and rotatory inertia increases with increasing coefficient of foundation and depth of the beam.

The expression for the bending moment can be obtained from (12), according to (3), with $c_4 = 0$. Considering (13), we find

$$M(x,\tau) = \frac{P}{2\pi c_3^{1/2}} \int_0^\infty \frac{\sqrt{z}\cos(x\sqrt{c_3 z})(1-\cos\omega_3\tau)}{1+z^2}\,dz \qquad \left(\omega_3 = \sqrt{\frac{1+z^2}{1+1.305\,z\gamma}}\right) \tag{15}$$

The effect of shear deformation and rotatory inertia is most significant in the case of short-time loading.

We shall compare the results of the classical theory of vibration of beams and the results of the wave theory for a loading in the form of a rectangular impulse with time of duration t_0. We shall consider the deflections and bending moments at the point of action of the impulse. We have for this case

$$y(0,\tau) = \frac{Pc_5^2}{2\pi c_3^{3/2}} \int_0^\infty \frac{(1+\gamma z)\,[\cos\omega_2(\tau-\tau_0)-\cos\omega_2\tau]}{\sqrt{z}\,(1+\gamma z+z^2)}\,dz \tag{16}$$

$$M(0,\tau) = \frac{P}{2\pi c_3^{1/2}} \int_0^\infty \frac{\sqrt{z}\,[\cos\omega_3(\tau-\tau_0)-\cos\omega_3\tau]}{1+z^2}\,dz \tag{17}$$

Expression (17) gives the results which are practically identical with those of the classical theory, even for large values of γ (0.05 to 0.07). The integral (16) was evaluated for $r_0 = 0.1$ by the use of Legendre polynomials and mechanical integration. In this, the weight function was assumed $p(z) = z^{-1/2}$, the interval of integration was assumed [0,8] and then reduced to [0,1]. The orthogonal systems of functions of weight $z^{-1/2}$ in the interval [0,1], are the polynomials $P_{2n}(\sqrt{z})$. The values of the roots and integral coefficients for these polynomials are given in [2]. In the figure, the solid line shows the deflection calculated on the basis of the classical theory, and the broken line shows the deflection obtained from Expression (16) for $\gamma = 0.065$.

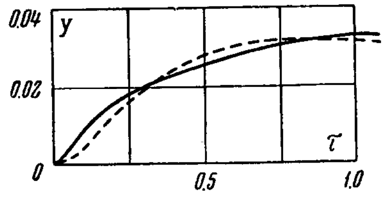

Fig. 1.

Considerable discrepancies can be noted in the initial stage, and both lines almost coincide as they approach the maximum value.

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